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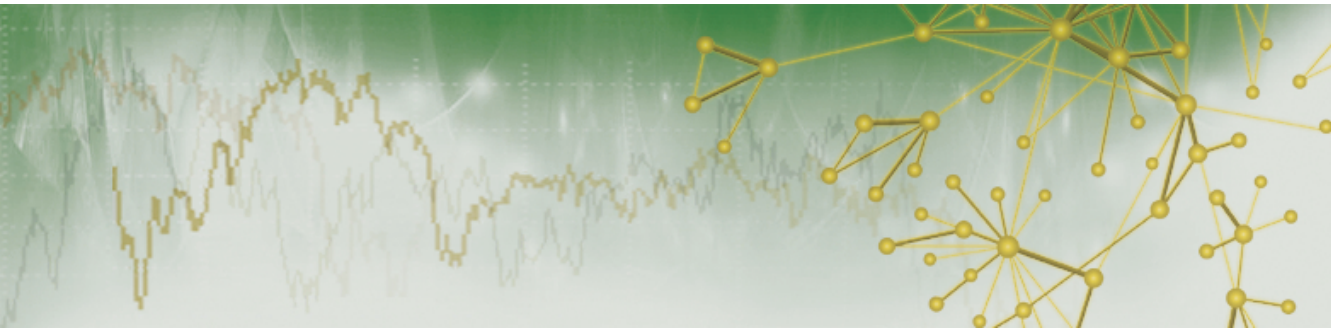
January 2009 REPORT A07

COMPLEX NETWORKS AND SPECTRAL METHODS: AN ECONOPHYSICS APPROACH TO EQUITY MARKETS

Tapio Heimo



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Tapio Heimo

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Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Biomedical Engineering and Computational Science

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Supervisor	Prof. Kimmo Kaski
Instructor	Dr. Jari Saramäki

Abstract

In analysing the top-level structure of financial markets rough simplifications are often needed due to their high degree of complexity. The framework of complex networks provides a modern way to do this effectively. In this approach, a market is seen as a network, such that the nodes of the network correspond to market participants and the links reflect the dynamic interactions between them.

This thesis contributes to the analysis of equity returns from the network point of view. By using empirical data on stock exchanges, we show how the general problem of extracting information from correlated time series can be formulated as a network problem. Then, we apply and develop network methods in order to analyse this problem. The purpose of the research is two folded. On the one hand, the goal is to extract new information on financial markets. On the other hand financial markets are seen as a data-rich test-bed for developing new methods and theories of more general interest.

In addition to network methods, correlations between equity returns are analysed on the basis of the spectral properties of the correlation matrix. We show how these two approaches can be combined to study the top-level structure of equity markets, and believe that the results are of very general interest as well. Most emphasis is given to the analysis of the information content of correlation matrices of equity returns.

Keywords	complex networks, spectral methods, equity markets, correlation matrices
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Vastaväittäjä(t)	prof. Peter Richmond
Työn valvoja	prof. Kimmo Kaski
Työn ohjaaja	TkT Jari Saramäki

Tiivistelmä

Rahoitusmarkkinoiden kompleksisuudesta johtuen niiden ylätason rakennetta tutkittaessa on tehtävä huomattavia yksinkertaistuksia. Markkinoiden kuvaaminen kompleksisena verkkona on moderni tapa tehdä nämä yksinkertaistukset. Tässä lähestymistavassa markkinoiden toimijat esitetään verkon solmuina ja toimijoiden väliset dynaamiset vuorovaikutukset linkkeinä solmujen välillä.

Väitöskirja edistää osakkeiden tuottojen analyysiä ja tutkimusta verkkonäkökulmasta. Osakkeiden tuottojen välisiin korrelaatioihin sisältyvän informaation suodattamiseen liittyvä ongelma kuvataan verkko-ongelmaksi, jota tutkitaan soveltamalla ja kehittämällä verkkopohjaisia menetelmiä. Tutkimuksen tavoitteet voidaan jakaa kahteen osaan. Toisaalta tavoitteena on saada uutta informaatiota rahoitusmarkkinoista, kun taas toisaalta markkinoita kuvaavaa dataa käytetään yleisten verkostomenetelmien ja -teorioiden kehittämiseen ja testaamiseen.

Verkkomenetelmien lisäksi osaketuottojen korrelaatioita analysoidaan myös niiden muodostaman korrelaatiomatriisin spektrin avulla. Väitöskirjassa osoitetaan, kuinka nämä kaksi lähestymistapaa voidaan yhdistää osakemarkkinoiden ylätason rakennetta tutkittaessa. Tuloksia voidaan erittäin todennäköisesti hyödyntää yleisesti myös muilla tieteenaloilla. Eniten painoarvoa väitöskirjassa on annettu osaketuottojen korrelaatioiden informaatiosisällön analysoimiselle.

Asiasanat	kompleksiset verkot, spektraalit menetelmät, osakemarkkinat, korrelaatiomatriisit		
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Preface

This dissertation was completed in the Centre of Excellence in Computational Complex Systems Research at Helsinki University of Technology and it concludes my Doctor of Science degree. Financially, this work was made possible by the very same Centre of Excellence, the European Commission through the NEST-Complexity project EDEN, a grant from EU COST Action P10 "Physics of Risk" and Nordea Bank. I have been fortunate to have the chance to work with many highly talented people and hope that some of their brilliance has been transferred to these pages together with my enthusiasm.

First and foremost, I wish to express my sincere gratitude to my supervisor Professor Kimmo Kaski for his continual encouragement and support as well as for giving me the chance to work in this fascinating field. I have been privileged to have Dr. Jari Saramäki as my instructor. Without his contribution and guidance this thesis would not have seen the daylight. In the early stages of this work, I had the opportunity to collaborate and share my ideas with Dr. Jukka-Pekka Onnela, whose work is the main reason, why I chose this field of research. My ex-roommate Jussi Kumpula shared my passion for analytical derivations. He also had the questionable honour to help me with many practical issues related to submitting this thesis. I would also like to thank my other collaborators Professor János Kertész and Gergely Tibély for their hospitality during my visit in Budapest and for their effort in our joint projects.

Besides people directly involved in my research, many colleagues in LCE/BECS contributed to this work. Especially, I wish to mention Dr. Riku Linna, Mikko Kivelä, Riitta Toivonen, Jaakko Niemi and Jenni Hulkkonen with whom I have had many lively discussions, sometimes on, but usually off the topic.

Finally, I wish to thank my family and friends. My language skills are insufficient to adequately describe the value of your unconditional support and endless patience with me.

Stockholm, 30th September 2008

Tapio Heimo

List of publications

This thesis consists of an overview and of the following publications, which are referred to in the text by their Roman numerals.

I T. Heimo, J.Saramäki, J.-P. Onnela, K. Kaski, "Spectral and network methods in the analysis of correlation matrices of stock returns", *Physica A* **383**, 147 (2007).

II T. Heimo, G. Tibély, J.Saramäki, K. Kaski, J. Kertész, "Spectral methods and cluster structure in correlation-based networks", *Physica A* **387**, 5930 (2008).

III T. Heimo, J.M. Kumpula, K. Kaski, J. Saramäki, "Detecting modules in dense weighted networks with the Potts method", *Journal of Statistical Mechanics*, P08007 (2008).

IV T. Heimo, J.Saramäki, K. Kaski, "Maximal spanning trees, asset graphs and random matrix denoising in the analysis of dynamics of financial networks", *Physica A* **388**, 145 (2009).

Author's contribution

The research reported in this thesis is a result of collaboration between Tapio Heimo, the author of this thesis, and the other authors of the included publications. As the primary author of all the publications, he is responsible for their written material and had a central role in the initiation of the research lines. In Publications (I,IV) he implemented all computer programs and performed the numerical analysis himself, whereas in Publications (II,III) he implemented the computer programs, performed the numerical analysis and derived the theoretical results together with the co-authors. In all the publications, the co-authors contributed in developing the ideas and research lines, in the analysis of the results, and in preparing the written material.

Contents

Chapter 1

Introduction

After the set up of world's first electronic stock market NASDAQ in 1971, financial markets grew rapidly to one of the largest and most data-rich socio-economic systems. This and perhaps the scarcity of research funding in physics after the end of the cold war, inspired the statistical physics community to use their expertise to analyse and model financial markets. In the business world, financial institutions started to make offers to holders of PhDs in physics, whereas in the academic world, a new interdisciplinary field of research, *econophysics*, was initiated. In typical econophysics studies ideas and methods are "borrowed" from statistical mechanics in order to tackle problems in finance or economics. Characteristic to these studies is a strong emphasis on empirical analysis, which is sometimes overlooked in the more traditional mathematical finance.

Prior to the 1990s, very few physicists did any professional research associated with financial or economic systems. However, some well known examples show that they were never immune to the temptations of finacial markets. Sir Isaac Newton's unsuccessful predictions of stock prices led to a terrible loss of 20000 pounds in 1720[1]. Carl Friedrich Gauss, on the other hand, managed the Göttingen University widow's fund very succesfully in the 1840s and went on to make a significant fortune through investments in corporate bonds. It was evident, long before the emergence of econophysics, that the fields of physics and economics had a lot to offer to each other. Physicists admit that the power-law distribution was first used by Pareto in his studies of wealth of individuals (1) and that the first mathematical formalization of a random walk and the formulation of the Chapman-Kolmogorov condition for Markovian processes by Bachelier (2) were motivated by observations of stock prices. On the other hand, it can be argued that mathematics entered economics through physics. Economists like Walras, Jevons, Fisher and Pareto used the formalism of physics, replacing material points by eco-

[1] After this, Newton gave the famous comment: *"I can calculate the motions of the heavenly bodies, but not the madness of people"*.

nomic agents and using the analogs of potential energy and principle of minimal action (3; 4).

Modern financial markets are quite far from the ones that Newton speculated. They can perhaps be best understood as a multi-agent game, in which millions of traders repeatedly compete against each other. The resulting interactions between market participants determine the collective opinion of the markets on the value of a company, reflected by the price of its *stock*. The underlying interactions cannot typically be measured directly, but since their manifestations in the markets are visible, a great deal of information about them is stored in stock price time series. Unfortunately, these time series also contain a lot of *noise*, which means that advanced methods are needed to filter real information. This is where the econophysicists step in.

In addition to econophysics, the late 1990s witnessed the emergence of another interdisciplinary field known as *network science*. This field of research relies on the assumption that the relevant information on *complex systems* can be captured with relatively simple objects, called *networks* or *graphs*. As in econophysics, most of the tools and ideas have been taken from the field of statistical mechanics. Financial markets serve as a good example of a complex system that can be approached with the network framework. This was first realized in 1999 by Mantegna, who was able to extract information from stock return time series with a very simple network-based method (5).

From the point of view of this thesis, the year 1999 was very significant. In addition to Mantegna's observations, it was discovered that the spectrum of empirical financial correlation matrices has a remarkable structure: all the information present in these matrices seems to be stored in a few eigenpairs (6; 7). Consequently, it was suggested that these "information carrying" eigenpairs could be used in identifying strongly correlated clusters of companies. In few years, the identification of clusters had become a focal topic in the network community though under the name *module detection*. This marked the starting point for the research presented in this thesis.

The purpose of this thesis is two-fold. On one hand, existing tools and theories are developed and applied to gain further insight on the structure of financial markets. Related to this, a lot of effort is made in order to combine the network and spectral based approaches for analysing financial correlation matrices. On the other hand, financial markets are considered as a data-rich "test-bed", on which new ideas and methods of more general interest can be developed. More specifically, many interesting general results for dense weighted networks are derived and empirically verified based on the properties of financial correlation matrices.

The structure of the overview part of the thesis is as follows. Chapter 2 reviews the elementary theory behind correlation matrices of asset returns and introduces how their information content can be analysed by using eigevalues and eigenvectors. The network approach to complex systems is presented in Chapter 3 together

with applications to analysis of equity markets. Chapter 3 also briefly discusses the correspondence between the spectral and network methods from the point of view of module detection. Summary and discussion are presented in Chapter 4.

Chapter 2

Correlation matrices of asset returns

This chapter provides a brief introduction to the analysis of correlation matrices of asset returns. The topic is of huge interest among financial practitioners, since the correlations between the components of a portfolio serve as main inputs in both *portfolio optimisation* and *risk management*. In the "classic" *modern portfolio theory* by Markowitz (8) risk is quantified by the variance of return, which is obviously directly related to the correlation matrix of the portfolio. More sophisticated approaches naturally exist, but still, almost irrespective of how one defines risk, the correlations between the components of a portfolio are the main inputs in the risk quantification. On top of this, correlation matrices are of interest because they can be used in creating a top-level description of the markets as a whole, as we shall see in Chapter 3.

This chapter starts with a short review of some elementary concepts and terminology related to asset returns. Then, we move to the analysis of correlation matrices of equity returns giving special emphasis to the information content of their eigenvalues and eigenvectors. Finally, we review how the spectral decomposition can be used to denoise financial correlation matrices. For a more comprehensive review on the econophysics approach to finance, see (4; 9; 10; 11).

2.1 Elementary theory of asset returns

The most fundamental observable in financial markets is the *price* of an asset, denoted here by $P(t)$. As time evolves, the prices of assets usually fluctuate and the change in price over some period is usually referred to as the *absolute return*[1].

[1]Here, stock splits, dividends and other accumulated cash flows are ignored for simplicity. They can be straightforwardly allowed for and the procedure can be found in almost any serious finance textbook. See *e.g.* (12)

From the point of view of an investor, however, a much more interesting quantity is the *arithmetic return*, usually denoted by R and defined as the percentage change in the price between two points in time

$$R(t_1, t_2) = \frac{P(t_2) - P(t_1)}{P(t_1)}, \tag{2.1}$$

or the *logarithmic return*, defined as

$$r(t_1, t_2) = \log \frac{P(t_2)}{P(t_1)}. \tag{2.2}$$

Logarithmic returns for longer periods of time can clearly be obtained by summing up logarithmic returns from shorter periods, when the dimension of $P(t)$ is left out. A very popular assumption is that over very short time periods, logarithmic returns are independent random variables, which means, according to the *central limit theorem*[2], that over longer time periods, say one day, they are normally distributed. This, together with symmetricity, is the main reason, why logarithmic return is the most widely used variable in mathematical finance and econophysics. However, it is well known that in practice distributions of logarithmic returns of assets have very fat tails compared to the normal distribution. Better fits can be obtained with *Lévy distributions* (see *e.g.* (9; 10; 13)) and so called *q-Gaussians* (see *e.g.* (14)).

An important issue when analysing asset prices is the way one chooses to measure time. Perhaps the most natural candidates for the time scale are the *physical time* and the *trading time*, defined as the time that elapses during open market hours. Out of these, the latter one is usually chosen, because the price fluctuations are to a large extent caused by the trading itself (12). In the analysis of high-frequency data, time is sometimes defined in terms of the number of transactions due to the fact that the market activity is far from uniform during the trading hours.

The data sets studied in this thesis consist of daily closing prices of equities, and therefore time is defined in terms of trading time such that the unit is one trading day. Here, we only deal with logarithmic returns, which we often refer to as *returns*.

2.2 Construction of correlation matrices

When estimating the (historical) return distribution of an asset, the first steps are to estimate the mean m and the variance σ^2. A typical assumption is that the

[2]The central limit theorem, interpreted loosely, states that large sums of independent identically distributed random variables of finite variance are normally distributed, except for the tails of the distribution.

evolution of the distribution is slow compared to the length of the time series used, which allows one to use the estimators

$$m = \langle \mathbf{r} \rangle, \qquad \sigma^2 = \langle (\mathbf{r} - \langle \mathbf{r} \rangle)^2 \rangle, \tag{2.3}$$

where $\mathbf{r}$ denotes the return time series used and angular brackets denote the (simple, non-weighted) average[3]. If the distribution evolves significantly as a function of time, exponential weighting or more advanced methods can be used (see *e.g.* (10) or (15)).

To be able to estimate the return distribution of a *portfolio* of assets, one needs to estimate the correlation coefficient of the returns between each pair of assets, *i.e.*, the correlation matrix of the returns. The estimator corresponding to Eq. 2.3 and used throughout this thesis is

$$\boldsymbol{C}_{ij} = \frac{\langle \mathbf{r}_i \mathbf{r}_j \rangle - \langle \mathbf{r}_i \rangle \langle \mathbf{r}_j \rangle}{\sqrt{[\langle \mathbf{r}_i{}^2 \rangle - \langle \mathbf{r}_i \rangle^2][\langle \mathbf{r}_j{}^2 \rangle - \langle \mathbf{r}_j \rangle^2]}}, \tag{2.4}$$

where $\mathbf{r}_i$ is a vector containing the logarithmic returns of asset i and $\langle \mathbf{r}_i \mathbf{r}_j \rangle$ denotes the average of the product of the returns of assets i and j over time[4]. If we set

$$\tilde{\mathbf{r}}_i(t) = \frac{\mathbf{r}_i(t) - \langle \mathbf{r}_i \rangle}{\sqrt{\langle \mathbf{r}_i{}^2 \rangle - \langle \mathbf{r}_i \rangle^2}}, \tag{2.5}$$

i.e., rescale the return vectors to have a zero mean and unit variance, we can rewrite Eq. 2.4 as

$$\boldsymbol{C}_{ij} = \langle \tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_j \rangle, \tag{2.6}$$

or in the matrix form as

$$\boldsymbol{C} = \frac{1}{T} \boldsymbol{M} \boldsymbol{M}^t, \tag{2.7}$$

where $\boldsymbol{M} = [\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots, \tilde{\mathbf{r}}_N]^t$ and T is the length of the time series.

As already discussed, the properties of correlation matrices are of wide interest. A bit surprisingly however, their empirical determination has turned out to be very difficult. Related to this, it was observed about a decade ago that correlation matrices of equity returns share many properties with purely random matrices. This is the topic of the next section.

2.3 Information vs. noise in financial correlation matrices

As mentioned above, the empirical determination of correlation matrices of asset returns has turned out to be very challenging. One major difficulty is the noisiness of the matrix. If a portfolio consists of N assets, its correlation matrix has

[3] If $\mathbf{r}$ has T components, $\langle \mathbf{r} \rangle = 1/T \cdot \sum_{t=1}^{T} \mathbf{r}(t)$

[4] Mathematically put, $\langle \mathbf{r}_i \mathbf{r}_j \rangle = 1/T \cdot \sum_{t=1}^{T} \mathbf{r}_i(t) \mathbf{r}_j(t)$

effectively $N(N-1)/2$ entries. Usually the length of the time series used in the estimation is not very large compared to N, as one assumes that the distribution of returns is constant during the time window used and the validity of this assumption clearly suffers if the length of the time window is increased. Therefore the number of data points is not very large compared to the number of matrix entries, a natural consequence of which is a noisy matrix. In their seminal work Plerou *et al.* (7) and Laloux *et al.* (6) discovered that the spectrum of correlation matrices of asset returns is very close to the spectrum of *Wishart matrices*. The Wishart matrix $W(N,T,\sigma)$ is a symmetric positive semi-definite random matrix[5] of size $N \times N$ defined by

$$W(N,T,\sigma) = AA^t, \tag{2.8}$$

where A is a $N \times T$ matrix with independent identically distributed Gaussian entries with zero mean and finite variance σ^2. The *eigenvalue density* of $W(N,T,\sigma)$, defined as

$$\rho(\lambda) = \frac{1}{N}\frac{dn(\lambda)}{d\lambda}, \tag{2.9}$$

where $n(\lambda)$ is the number of eigenvalues smaller than λ, is known in the limit $N,T \to \infty$, $T/N \geq 1$ fixed. After "normalizing" the matrix with $1/T$ as in Eq. 2.7 it can be written (17; 18) as

$$\rho(\lambda) = \begin{cases} \frac{T/N}{2\pi\sigma^2}\frac{\sqrt{(\lambda_{\max}-\lambda)(\lambda-\lambda_{\min})}}{\lambda} & \text{if } \lambda_{\min} \leq \lambda \leq \lambda_{\max} \\ 0 & \text{else} \end{cases} \tag{2.10}$$

$$\lambda_{\max/\min} = \sigma^2\left(1 \pm \sqrt{N/T}\right)^2, \tag{2.11}$$

Results corresponding to Eqs. 2.10 and 2.11 have been derived *e.g.* for the exponentially weighted moving averages based approach (10; 19; 20; 21) and for return distributions with power-law tails (22; 23) as well.

Plerou *et al.* and Laloux *et al.* studied the spectrum of empirical correlation matrices with the null hypothesis of completely random matrix (Wishart) and discovered that around 95% of the eigenvalues of empirical correlation matrices fall in the region where Eq. 2.10 applies (see Fig. 2.1). From this they concluded that only 5% of the eigenpairs carry information and verified this by comparing the nearest-neighbour and the next-nearest-neighbour spacings of the eigenvalues (7) to the analytical results for the *Gaussian Orthogonal Ensemble* (see *e.g.* (16)) and the distribution of the eigenvector components to the maximal entropy distribution (Porter-Thomas) (6; 7). The results have been empirically verified for all the major stock exchanges and extended in *e.g.* (24; 25; 26; 27; 28; 29; 30; 31) as well as in Publications (I,II) with the conclusion that after ranking the eigenvalues in decreasing order, the eigenpairs of correlation matrices of equity returns can be classified as follows:

[5] For a review of random matrix theory, see (16)

1. The lowest ranking, *i.e.*, smallest eigenvalues, do not belong to the random part of the spectrum. The corresponding eigenvectors are highly localized, *i.e.*, only a few assets contribute to them.

2. The next lowest ranking eigenvalues (about 90-95 % of all eigenvalues) form the "bulk" of the spectrum. They, or at least most of them, correspond to noise and are well described by random matrix theory.

3. The highest ranking eigenvalue is well separated from the bulk and corresponds to the whole market as it is practically directly proportional to the mean of the correlations and the corresponding eigenvector has roughly equal components.

4. The next highest ranking eigenvalues and the corresponding eigenvectors also carry information about the real correlations and are related to clusters of strongly interacting assets. The randomness present in these eigenpairs increases rapidly together with decreasing rank (on average).

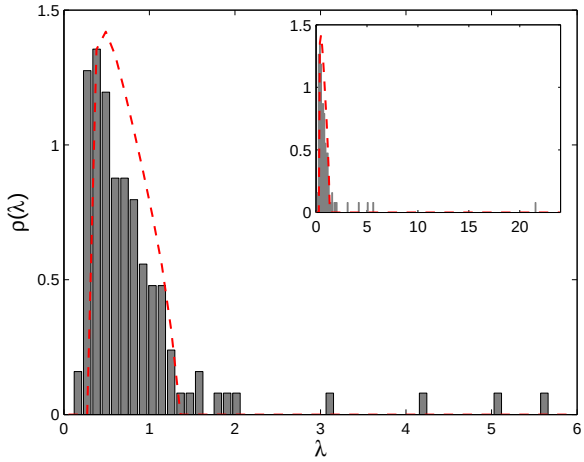

Figure 2.1: Eigenvalue density for an empirical correlation matrix of $N = 116$ NYSE-traded stocks (bars) and the theoretical curve for Wishart matrices (dashed line). The width of the time window used is $T = 1000$ trading days and the parameter σ^2 has been adjusted visually to account for the non-random part of the spectrum (see section 2.5 for discussion).

2.4 Non-random eigenpairs

As stated in the previous section, the eigenpairs corresponding to the highest eigenvalues of financial correlation matrices are far from being random. The highest eigenvalue is always clearly separated from the rest of the spectrum and the corresponding eigenvector, usually referred to as the *market eigenvector*, is typically interpreted to be representative of the whole market (25). In (32) and in Publication II it is shown that according to first order perturbation theory the components of this eigenvector are proportional to the corresponding row (or column) sums of the correlation matrix and that the highest eigenvalue itself is proportional to the average off-diagonal matrix element. This is illustrated in Fig. 2.2.

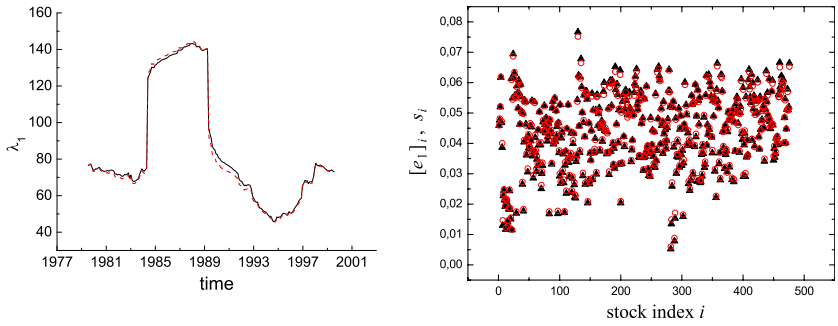

Figure 2.2: Left: The first eigenvalue (solid line) of the correlation matrix and rescaled mean correlation (dashed line) as functions of time for a data set of 476 NYSE-traded stocks. The outstanding plateau is a consequence of Black Monday, a large market crash on October 19, 1987 and its width corresponds to the width of the time window used ($T = 1000$ trading days). The horizontal axis is chosen such that it corresponds to the center of the time window. Right: The components of the first eigenvector and rescaled row sums of the correlation matrix for a chosen time window.

The next highest ranking eigenpairs also contain information, but the randomness increases rapidly together with decreasing rank (on average), meaning that there is no strict boundary between the random and non-random parts of the spectrum. As first discovered by Laloux *et al.* (6) and Plerou *et al.* (7), many of the high ranking eigenvectors are localised on business sectors or industries (see Fig. 2.3). However, based on the work presented in Publication II it seems evident that the cluster structure of the correlation matrix cannot be easily deduced from its eigenvectors. Most of the information about this structure can only be found by combining information from different eigenvectors, as first suggested in (25), but there is no rule to tell which combination of the eigenvectors should be taken.

Therefore, the extraction of the cluster structure based on the eigenvectors seems to be a too formidable task without *a priori* information.

2.5 Random matrix denoising

The fact that most eigenpairs of correlation matrices of asset returns correspond to noise can be rather easily used in noise filtering. The idea of the method, known as *random matrix denoising* or *eigenvalue cleaning* and again first suggested in (6), is to replace the eigenvalues corresponding to noise with a unique eigenvalue such that the trace of the matrix is preserved. The only not straightforward step of the method is to decide, which eigenpairs are considered as corresponding to noise. Ideally, this should be done based on a thorough study of the eigenpairs. However, the most popular way is to use λ_{max} and λ_{min} (or only λ_{max}) defined in Eq. (2.11). With this approach, one has to take into account that not all the eigenvalues correspond to noise and therefore the value of σ^2 has to be accordingly modified. Quite often, only the contribution of the largest eigenvalue λ_1 is taken into account, which leads to $\sigma^2 = 1 - \lambda_1/N$. A more sophisticated way is to fit Eq. (2.10) to the observed distribution of eigenvalues using σ^2 as an adjustable parameter, as suggested in (6), and then use the optimal value of σ^2. A feature not usually taken into account in this approach is that for finite N the theoretical borders $\lambda_{\text{max/min}}$ become blurred, *i.e.*, the probability of eigenvalues outside the interval $[\lambda_{\text{min}}, \lambda_{\text{max}}]$ is no longer equal to zero. Fortunately, through the work by Kim *et al.* (32), we know that the structure of the denoised matrix is not very sensitive to the inclusion or exclusion of a few eigenpairs around λ_{max}.

After the set of information carrying eigenpairs has been determined, the denoising procedure continues as follows (19; 33; 34). One starts by expanding the ordinary correlation matrix $\boldsymbol{C}$ as

$$\boldsymbol{C} = \sum_{i=1}^{N} \lambda_i |\lambda_i\rangle\langle\lambda_i| = \sum_{i \in I} \lambda_i |\lambda_i\rangle\langle\lambda_i| + \sum_{i \in I_r} \lambda_i |\lambda_i\rangle\langle\lambda_i|, \tag{2.12}$$

where I denotes the index set of the information carrying eigenpairs and I_r the index set of eigenpairs corresponding to noise. Now, by defining

$$\xi = \frac{\text{Tr}\, \boldsymbol{C} - \sum_{i \in I} \lambda_i}{|I_r|} = \frac{\sum_{i \in I_r} \lambda_i}{|I_r|}, \tag{2.13}$$

where $|\cdot|$ denotes the number of elements in the set, the denoised matrix can be written as

$$\tilde{\boldsymbol{C}} = \sum_{i \in I} \lambda_i |\lambda_i\rangle\langle\lambda_i| + \sum_{i \in I_r} \xi |\lambda_i\rangle\langle\lambda_i|. \tag{2.14}$$

Clearly, the diagonal elements of $\tilde{\boldsymbol{C}}$ are no longer equal to unity, although the trace of the matrix is preserved. Therefore it is perhaps more natural to consider $\boldsymbol{C}$ as the covariance matrix of the time series rescaled to have a unit variance and $\tilde{\boldsymbol{C}}$ as the corresponding denoised covariance matrix. Naturally, this covariance matrix can be further transformed into a correlation matrix by setting $\hat{\boldsymbol{C}}_{ij} = \tilde{\boldsymbol{C}}_{ij}/\sqrt{\tilde{\boldsymbol{C}}_{ii}\tilde{\boldsymbol{C}}_{jj}}$.

The conservation of the trace is very important if the denoised matrix is used in portfolio optimization. Regardless of the method used, the covariances between the assets are closely related to the risk of a portfolio. If the trace is not preserved, there will be a bias in the risk estimation. However, if the correlation matrix is used in identifying groups of strongly correlating assets, the conservation of the trace is not crucial. Kim *et al.* (32) have suggested that setting ξ equal to zero in the above denoising process would yield matrices from which the modular structure could be easily identified with *thresholding analysis*. This is further investigated in Publication II and discussed in section 3.4.1.

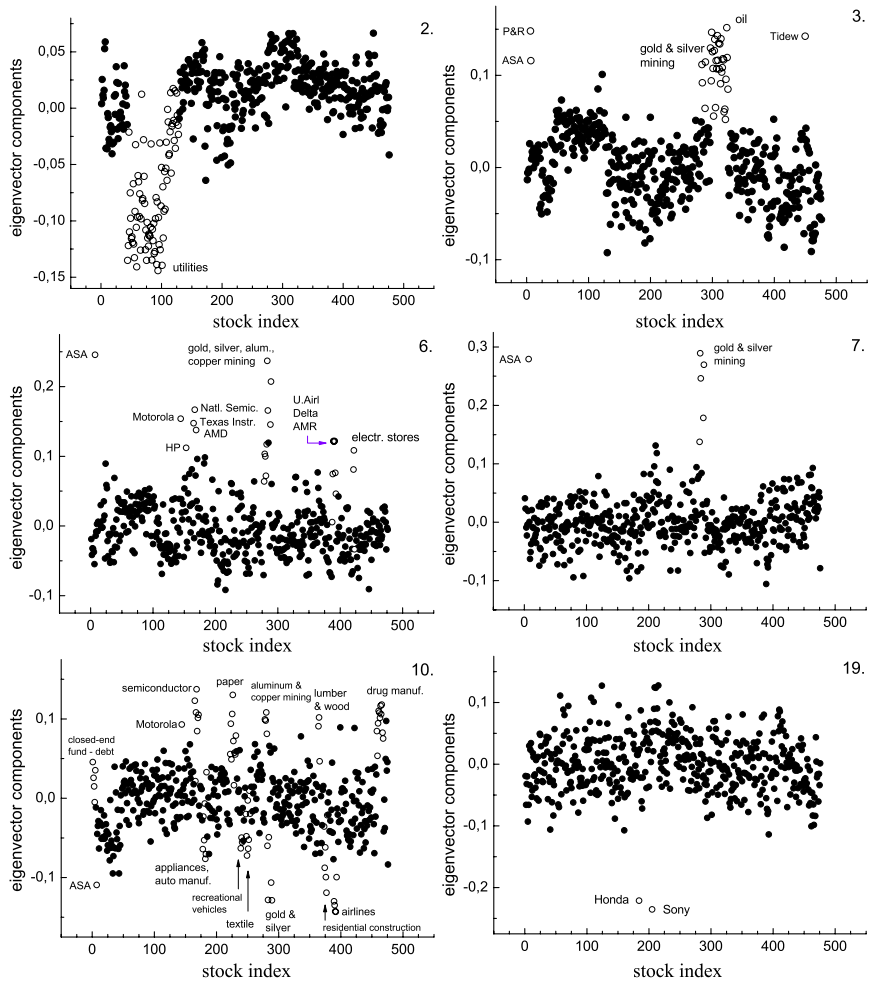

Figure 2.3: Component sizes of chosen eigenvectors for a data set of 476 NYSE-traded stocks over the period from 2-Jan-1980 to 31-Dec-1999 (T=5056). The number on the top right of each panel indicates the rank of the corresponding eigenvalue. Horizontal ordering is such that stocks belonging to same business sectors are next to each other and open symbols are used as a guide to the eye. The eigenvectors of ranks 2, 3 and 7 are localised relatively well to business sectors or industries. However, these modules stand out well only because of the chosen horizontal ordering of the companies. Without *a priori* information we would not be able to define boundaries for these modules. The eigenvector of rank 10 serves as a good example of an eigenvector localised to multiple modules. Clearly, without prior information, interpretation of this eigenvector is impossible. Surprisingly, the eigenvector of rank 19 is very far from being random even though the corresponding eigenvalue is already relatively small. A module consisting of the only Japanese companies in the data set, Sony and Honda, can easily be identified from this eigenvector.

Chapter 3

Network approach to complex systems

Complex systems are found from almost all fields of science. Characterictic to these systems is that they consist of a large number of interacting elements such that the emergent system-level properties cannot be easily understood based on the properties of the single constituents. Consider, *e.g.*, traffic jams, ant colonies, climate or human society. All these are good examples of systems where highly non-linear aggregate activity emerges from the individual actions of the elements. Financial markets serve as a solid example as well. They exhibit all the aforementioned properties, and since in addition, a huge amount of empirical data on them has been collected, they provide a valuable "test-bed" for developing new tools and theories of more general interest. Obviously, some people find the study of the markets itself quite "rewarding".

As complex systems cannot be understood without analysing them as a whole, one is forced to focus on a system-level description. In many cases this can be achieved by representing the system as a network of interacting elements. The last section of this chapter introduces how this framework can be used in the analysis of financial, or more specifically, equity markets. However, before that the chapter provides a brief overview of network theory in a more general context. Emphasis is naturally given to the areas closely related to the publication part of this thesis. For a comprehensive review of the network framework, see (35; 36; 37; 38; 39; 40; 41).

3.1 Networks are everywhere

Loosely defined, a network, or a graph as the mathematicians prefer, is a collection of items with connections between them (see Fig. 3.1). The items are called *nodes* or *vertices* and the connections, usually representing some kind of interac-

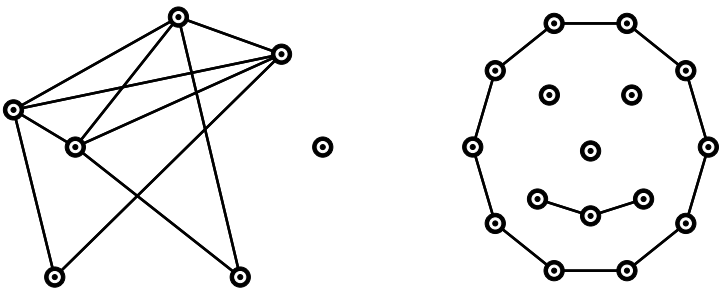

Figure 3.1: Left: A small network with seven nodes and ten links. Right: A small network with sixteen nodes and twelve links.

tions between these items, *links* or *edges*. During the recent years networks have proven to be a very efficent way to characterize and investigate a wide range of complex systems from the biochemistry of our cells (42; 43; 44; 45) to the Internet (40; 46; 47; 48), from patterns of interactions between individuals (49; 50; 51) to relations between word classes in a thesaurus (52; 53; 54). The key reason for the success of the network approach probably lies in its simplicity. One is able to focus on the system-level description instead of often very complicated details. Sometimes the pure topology of a network carries enough information about the system under study, *i.e.*, it is enough to consider the links as binary, such that each link either exists or not. However, valuable information is often lost if the interaction strengths are not taken into account. Because of this, the study of *weighted networks* (55), in which a scalar weight representing the associated interaction strength is assigned to each link, has also received a lot of attention.

The study of networks was initiated by the Swiss mathematician Leonhard Euler, who solved the problem known as the *Seven Bridges of Königsberg* (see Fig. 3.2) in 1736 using a network based approach (56). The next seminal advancement was the initiation and development of Erdös-Rényi (ER) graphs and what is known as *random graph theory* over 200 years later in the 1950s and 1960s by Solomonoff, Rapoport, Erdös and Rényi (57; 58; 59; 60). The (most famous) ER-graph $G(N, p)$ is a network of N nodes, in which each link exist with a probability p. Many properties of $G(N, p)$ are exactly solvable in the limit of large size[1], which explains a lot of the attention ER-graphs have been given and is one of the main reasons why they were used in the modelling of real-world graphs for several decades. Later, these graphs have been most notably studied by

[1]Typically the limit is taken such that $z = p(N-1)$ is fixed. Most results can be generalized for another ER-graph $G(N, m)$, consisting of N nodes and exactly m random links. $G(N, m)$ provides a natural random reference for *thresholding analysis* discussed later in this thesis.

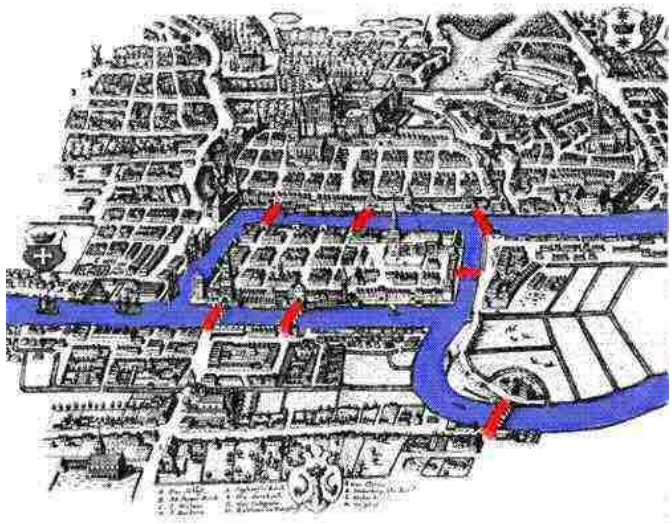

Figure 3.2: Map of Königsberg in Euler's time highlighting the bridges and the river Pregel. The problem is to decide whether it is possible to walk a route that crosses each bridge exactly once and returns to the starting point. Euler was able to show that this is not possible by formulating and proving the *planar graph theorem* (56).

Bollobás (61).

The increased availability of data on real-world systems together with increased computing power led to a substantial new interest in network research, especially among physicists, in the late 1990s. Before physicist, networks had been mainly used by social scientists to describe interactions between individuals, the typical size of the networks being tens or sometimes hundreds of nodes. In the studies carried out by physicists, it is not uncommon to encounter networks with millions of nodes. Much of the recent interest was triggered by the seminal work by Watts and Strogatz (62) and Barabási and Albert (63), who showed that real-world networks from diverse fields share certain universal properties which are far from those of ER-graphs. Most importantly, they were able to explain the emergence of these properties with simple models. After this, the field of network science has developed very rapidly, especially from the point of view of statistical mechanics. Recently, the focus of research has been shifting towards the functionality of networks, which in general cannot be explained by concentrating on single nodes or their immediate neighbourhoods. A related topic of high interest is the identification of *modules* discussed in section 3.3. First however, let us go through some basic terminology of network science.

3.2 Basic network terminology

This section briefly reviews the basic concepts and definitions that one encounters in the publication part of this thesis. A more detailed review can be found *e.g.* in (64).

As defined in the previous section, a network is a collection of items with connections between them. Mathematically put, $G = (V, E)$, where G is the network (or graph), V the set of nodes (or vertices) and $E \subset \{(v_i, v_j) | v_i, v_j \in V\}$ the set of links (or edges). In general the links can have a direction, *i.e.*, $(v_i, v_j) \neq (v_i, v_j)$ when $v_i \neq v_j$. However, this is not the case for the networks studied in this thesis and therefore we define all the links as *undirected*. Mathematically put, this means that we consider the links as unordered pairs so that $(v_i, v_j) = (v_j, v_i)$. In addition, we do not allow *self-edges* or *loops*, *i.e.*, links of the form (v_i, v_i). A network $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$ is called a *subgraph* of G and if $(v_i, v_j) \in E' \quad \forall v_i, v_j \in V'$, G' is a *clique*. If in the latter case $V' = V$, the network is said to be *full*. The *adjacency matrix* of the network is denoted by A and has entries $A_{ij} = 1$ if $(v_i, v_j) \in E$ and $A_{ij} = 0$ otherwise. Also, if $(v_i, v_j) \in E$, v_i and v_j are called *neighbours*. The set of neighbours of a node is called its *neighbourhood* and the size of this set k_i is the *degree* of the node. A node with degree one is a *leaf* and a node with degree zero an *isolate* or an *isolated node*. The clustering coefficient of node i is defined as $c_i = 2N^i_\Delta/[k_i(k_i - 1)]$, where N^i_Δ is the number of links between the neighbours of the node.

A *path* is a sequence of nodes $\{v_1, v_2, \ldots, v_n\}$ such that $(v_i, v_{i+1}) \in E$ for all $i \in [1, 2, \ldots, n-1]$ and its *length* is $n - 1$. The path between v_i and v_j that has the smallest possible length is called a *shortest path* or a *geodesic* and the length of this path is called the *distance*[2] between the nodes in question. The largest distance between any pair of nodes is called the *diameter* of the network. A path is a *circuit*[3] if $v_1 = v_n$ and a network without circuits is a forest. A maximal subset $\tilde{V}$ of V, such that there is a path between every pair of nodes in $\tilde{V}$, is called a *connected component* or sometimes *isolated component* or just *component*. If there is a $\tilde{V} = V$, the network is $connected$. A connected forest is called a $tree$.

Weighted networks are networks in which a number (usually a positive real number) is assigned to each edge, *i.e.*, the edges are of the form $((v_i, v_j), w_{ij})$, where $(v_i, v_j) \in E$ and $w_{ij} \in \mathbb{R}_+$. The matrix with entries $W_{ij} = w_{ij}$ if $(v_i, v_j) \in E$ and $W_{ij} = 0$ otherwise, is called the *weight matrix*. Sometimes, weighted networks are considered as full networks and the zero weight is used to denote the "absence" of a link. Clearly, this does not change the weight matrix and the *strength* of a node v_i can be defined as $s_i = \sum_j W_{ij}$ in both cases.

In *thresholding analysis* a weighted network is studied such that only a set

[2] This is not the only way the distance can be defined as we will see later in this thesis.

[3] Sometimes a circuit is called a cycle or even a loop.

fraction of strongest links are considered present. The resulting network, which can be considered as either binary or weighted, is characterized by a parameter p, which is the ratio of the number of present links to the number of all possible links. The *maximal spanning tree* of a weighted network is a (binary or weighted) tree connecting all the the nodes of the network such the sum of weights of the included links is maximized. Correspondingly, a similar tree in which the sum of weights is minimized is called the *minimal spanning tree*.

3.3 Modular structure of networks

3.3.1 What is a module?

As already mentioned, the focus of network research has recently been shifting towards "mesoscopic" properties of networks, *i.e.*, structures beyond the scale of single nodes or their immediate neighbourhoods. A lot of attention has been given to the study and detection of *modules* or *communities*, *i.e.*, groups of nodes with dense internal connections and weaker connections to the rest of the network. In the following, the most widely used methods for detecting modules in both binary and weighted networks are presented after a brief introduction to the topic. A few more advanced topics related to the publication part of this thesis are discussed as well.

Modules are usually related to functional units of the system under study, *e.g.*, companies operating in the same business sector in a financial network, circles of friends in a social network or pages related to similar topics in the World Wide Web. The structure of the modules often provides information on the roles of individual nodes. This is the main reason for the huge amount of attention their study has recently been given. However, even though new module detection methods are published frequently, a widely accepted definition of a module is yet to emerge. In general, module detection methods can be classified in global and local methods depending on whether a method and the incorporated definition of a module rely on global or local network properties. Global methods include the widely used *modularity optimization* methods introduced by Newman and Girvan (65) as well as the closely related q-state Potts method by Reichardt and Bornholdt (66; 67). Also, the category of global methods can be seen to include the recent information-theory based methods (68; 69) and the approaches based on *diffusion* (70; 71; 72), global centrality measures (73; 74) as well as the spectral properties of the weight matrix, discussed in Publications (I,II).

Good examples of local methods are the *clique percolation* method by Palla *et al.* (75) and a recent method by Lancichinetti *et al.* (76). Both methods have the advantage that they allow for *overlapping* modules, *i.e.*, a single node may belong to several modules at the same time, which is a very natural requirement in many cases. For instance, in a financial network a Japanese car company might belong

to a module of car companies as well as to a module of Japanese companies, or in a social network an individual might belong to a module consisting of his relatives as well as to a module formed by his friends (see Fig. 3.3 for a further example). Further advantages of the local methods are that they are often computationally more efficient than global ones and that local changes in the structure of the network do not change the modular structure in the distant parts of the network. The latter property is especially handy when studying evolving networks.

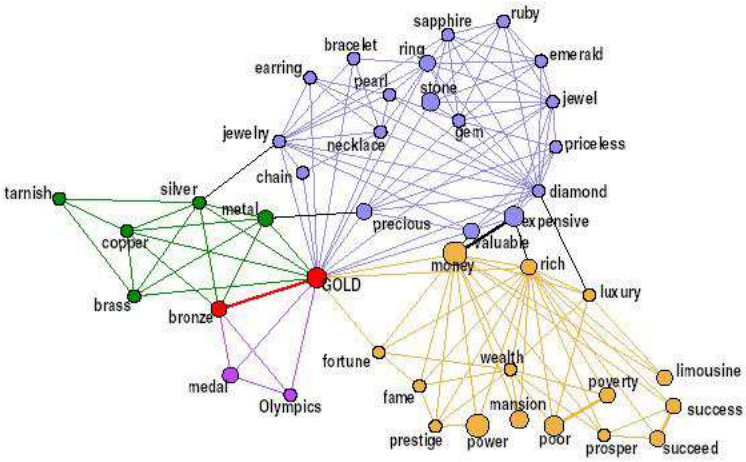

Figure 3.3: Modules of the word *gold* in a word association network. The purple module is related to Olympic medals, the green one consists of metals, the blue one can be associated to jewels and the yellow one is related to welfare. Figure published in (75), copyright (2005) Nature Publishing Group.

3.3.2 Modularity and the q-state Potts optimization

The *modularity* measure was introduced by Newman and Girvan (65) in 2004 to assess the quality of a given partition of a binary network into modules. Modularity optimization became almost immediately a very popular way to detect the modular structure of networks. It still continues to be that even though, due to the work by Fortunato and Barthélemy (77), it is known to have a *resolution limit*: small modules in large networks cannot be detected. The modularity Q can be written as

$$Q = \frac{1}{L}\sum_{m}(l_{mm} - [l_{mm}]), \tag{3.1}$$

where L is the number of links in the network, l_{mm} the number of links inside module m and $[l_{mm}]$ the expected number of these links in the *configuration*

model (78), *i.e.*, a random network with the same degree sequence as the original one. It has been shown that modularity optimization is an NP-complete problem (79). Therefore, approximative optimization methods, such as simulated annealing (80), greedy agglomeration (81; 82), extremal optimization (83) or spectral division (84; 85), have to be applied.

A more general approach based on the q-state Potts model was suggested by Reichardt and Bornholdt in 2004. In their method, known as the RB-method, the quality of a partition is measured by the *Hamiltonian*

$$\mathcal{H} = -\sum_m (l_{mm} - \gamma[l_{mm}]_{p_{ij}}). \tag{3.2}$$

The new features compared to the modularity are the adjustable parameter γ and the random reference p_{ij}, over which the expected number of links is taken. The random reference can be selected freely, but in practice, the configuration model is usually chosen in which case the method is equivalent to the modularity optimization when $\gamma = 1$. The resolution limit is present in the RB-method as well (86). The parameter γ can be used to tune the resolution, but there is no way to know which value of γ should be used without prior information about the structure of the network. Another, more practical issue related to the use of the RB-method is the minimization of the Hamiltonian. In principle, the same approximate methods as with the modularity can be used. In the publication part of this thesis simulated annealing (67; 80) was applied with single-spin and block flips as the elementary Monte-Carlo operations.

Even though the above methods were initially developed for binary networks, they were very soon generalized for weighted networks as well. This generalization, however, was done in a very straightforward way without a proper derivation of the random reference network. The weighted version of the RB-Hamiltonian can be written as

$$\mathcal{H}_w = -\sum_m (w_{mm} - \gamma[w_{mm}]_{p_{ij}}), \tag{3.3}$$

where w_{mm} is the sum of weights of the links inside module m and $[w_{mm}]_{p_{ij}}$ the expected value of this sum in the chosen random reference network. In Publication III, we suggest a weighted random reference closely related to the configuration model and derive the corresponding $[w_{mm}]_{p_{ij}}$.

Since the most natural representation of many real-world networks is a full or an almost full weighted network, it is a bit surprising that the performance of the RB-method with such networks had not been given any attention before the work presented in Publication III. For sparse weighted networks the resolution of the RB-method behaves similarily with the unweighted case. However, the theoretical results derived in Publication III for simple example networks (see Fig. 3.4 and Fig. 3.5) suggest that in large dense weighted networks the resolution of the RB-method does not decrease with increasing network size. In the same work, the

RB-method is also succesfully used in the analysis of a full financial network. We return to this in section 3.4.2.

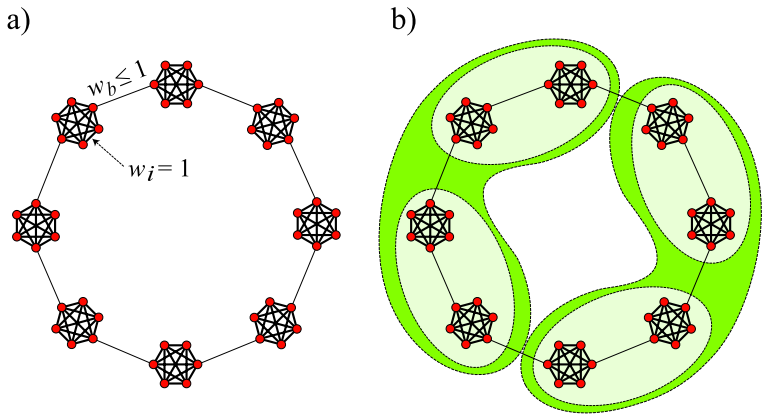

Figure 3.4: a) A ring-like network, consisting of N_b cliques, each containing of N_c nodes. Link weights w_i within modules equal unity, whereas modules are joined by links of weight $w_b \leq 1$. b) The weighted RB method can merge consecutive cliques to larger modules, depending on the values of the network parameters and the tuning parameter γ. The hierarchical structure is for illustrative purposes only. In general, the RB method does not yield hierarchical modules.

3.3.3 Clique percolation method

As defined in section 3.2, a clique is a full subgraph of a network. When one wants to emphasize that a clique consists of k nodes, it is usually called a k-clique. Thus, a 3-clique is synonymous to a triangle etc. In the famous paper by Palla *et al.* (75) the authors introduce how this concept can be used in module detection in a very efficient way. Their method, known as the clique percolation method, defines modules as sets of nodes belonging to adjacent k-cliques, *i.e.*, k-cliques sharing $k-1$ nodes. This is illustrated in Fig. 3.6. When applying the method, the choice of k allows one to explore the modular structure on various hierarchical levels. Other advantages of the method are that it is deterministic and allows for overlapping modules. However, in very sparse networks most nodes are not assigned to any module, which may sometimes be problematic. Most notable applications of the method include analysis of co-authorship, word-association and protein-interaction networks (75), detection of protein modules involved in cancer metastasis (87) and study of social group evolution (88).

In 2007 Farkas *et al.* (89) introduced a weighted version of the clique percolation method. The fundamental idea of the method is that only cliques with

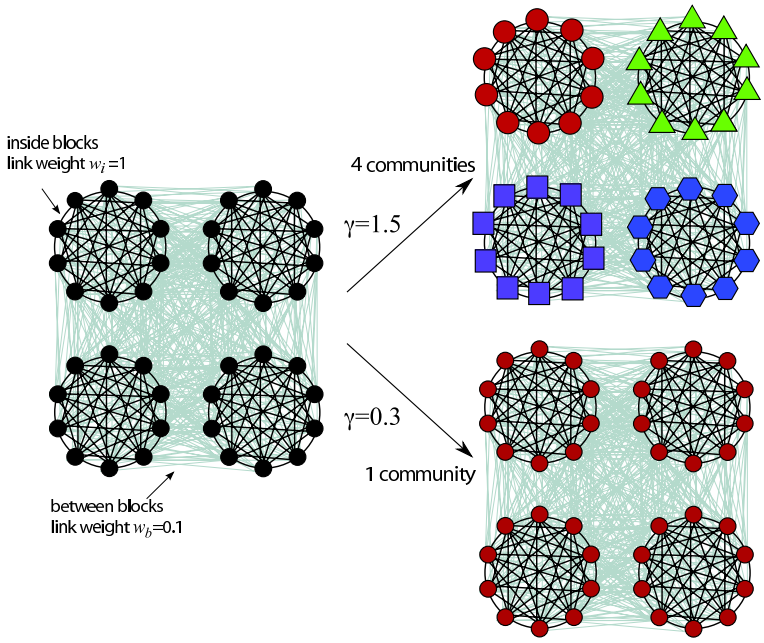

Figure 3.5: Left: A network consisting of $N_b = 4$ blocks each having $N_c = 10$ nodes. Links inside blocks have weight $w_i = 1$ and nodes in different blocks are connected with links of weight $w_b = 0.1$. The effect of the value of γ on the found modular structure is illustrated on the right. Large values yield the physical communities while for small values the communities appear as one large module. If the number of blocks N_b is large enough, the networks size does not affect the values of γ where merging takes place.

intensity (90) above a set intensity threshold I_c are considered as present, intensity being defined as the geometric mean of the links. In Publication I, another popular approach which utilizes thresholding analysis together with the ordinary clique percolation method was taken. Related to this approach, Kumpula *et al.* have recently introduced a fast algorithm, which allows one to perform clique percolation for multiple weight thresholds with a single run (91).

3.3.4 Multiresolution approach to module detection

Originally, the goal of module detection was to assign a module or modules to each node. However, the modular structure of a network can be far more complicated. The modules may occur on various levels, which may be hierarchical or not. When analysing the modular structure of such networks, methods incorporating a continuous parameter that can be used to tune the resolution of the

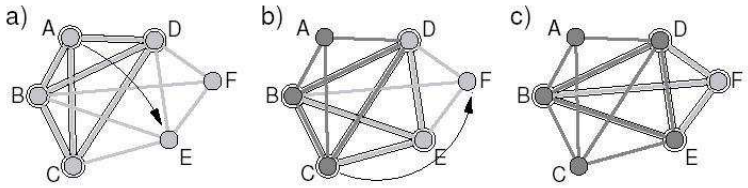

Figure 3.6: Illustration of the clique percolation method. A k-clique community is defined as a maximal set of nodes that can be reached by "rolling" a k-clique over adjacent k-cliques, *i.e.*, k-cliques sharing $k-1$ nodes. Panels a) and b) initial 4-clique ABCD rolls on to BCDE by releasing node A. Panels b) and c) the clique rolls on to BDEF by releasing node C. Figure published in (88), copyright (2007) Nature Publishing Group.

method are very useful. Examples of such parameters are the parameter γ of the RB-method, the intensity threshold I_c in the weighted clique percolation method, and the weight threshold p in thresholding analysis. In (92) the idea of *stable regions*, *i.e.*, ranges of the value of a tuning parameter where the obtained modular structure remains essentially stable, is introduced. Such regions can be expected to correspond to the most relevant levels of the modular structure. In Publications (I,II,IV) thresholding analysis is applied from this point of view, and in Publication III the approach is used with the RB-method and the multiresolution method by Arenas *et al.* (92).

3.4 Correlation-based financial networks

As discussed in the previous sections, networks provide an efficient way to represent and characterise complex systems with a large number of interacting elements. This section introduces how networks can and have been used in the analysis of the top-level structure and dynamics of the financial markets. Emphasis is again naturally given to the work related to the publication part of this thesis.

3.4.1 Asset tree and asset graph approaches

Mantegna was the first to study the clustering of companies using correlation-based networks. In his famous paper (5) he constructs the correlation matrix of stock returns by Eq. (2.4) and defines a *metric* to the space of assets by

$$d(i,j) = \sqrt{2(1-\boldsymbol{C}_{ij})}. \tag{3.4}$$

Mantegna discovered that the minimal spanning tree based on these distances contains a lot of economic information. Moreover, this method enables one to describe the hierarchical organization of the markets. Later, it has been shown that the trees produced this way are relatively robust (93; 94) and diverse equity markets have been studied with this approach with the conclusions that the most central nodes of the tree are mostly large conglomerates and financial institutions, whereas the branches often correspond to specific business sectors or industries (95; 96; 97; 98; 99). Naturally, the topic has also been approached with portfolio analysis in mind and it has been shown that assets belonging to the minimum-variance portfolio are located on the outer leaves of the tree (99; 100; 101). In addition to intramarket correlations, also intermarket correlations (102; 103; 104) as well as interest rates (105) and foreign exchange (106; 107) data have been studied with the method. It has also been suggested that *planar maximally filtered graphs* yield a natural extension to the tree approach (108; 109).

The tree approach used in this thesis and in the work by the author is equivalent to that of Mantegna. However, the trees are constructed directly from the correlation matrices, such that the sum of the correlation coefficients $\boldsymbol{C}_{ij}$ is maximized[4] and therefore they are called maximal spanning trees (MST). An illustration of such a tree is given in Fig. 3.7

The MST approach bears many similarities with the thresholding approach, introduced in the study of financial correlation matrices by Onnela *et al.* (99; 110; 111). Onnela is also the father of the names *asset tree* and *asset graph*, which he used to denote the MSTs and thresholded graphs based on asset return time series, respectively. With small values of the thresholding parameter p, *i.e.*, when only the strongest links are included, the clustering coefficient of the network is very high[5] and the strongest business sectors are seen as strong isolated components, corresponding to the branches of the MST. When p is increased these components start merging, but in Publication I it is shown that the modules can still be rather easily identified with the clique percolation method. This is illustrated in Fig. 3.8 together with the localisation of chosen eigenvectors. The correspondence between the eigenvectors and the networks observed in Publication I is remarkable and suggests that asset trees and asset graphs can be used in identifying information carrying eigenpairs of a correlation matrix. Such a comparison also serves as a great "sanity-check" that both methods yield sensible results. A check like this can be very valuable in the analysis of diverse systems, especially when no *a priori* information about the structure is available.

The time dependence of market correlations was first studied from the network point of view by Onnela *et al.* (99; 100). They were able to define many useful

[4]Since $d(i,j)$ is a decreasing function of $\boldsymbol{C}_{ij}$ the trees obtained with the two approaches are equivalent

[5]Compared to the natural Erdös-Rényi reference $G(N,m)$.

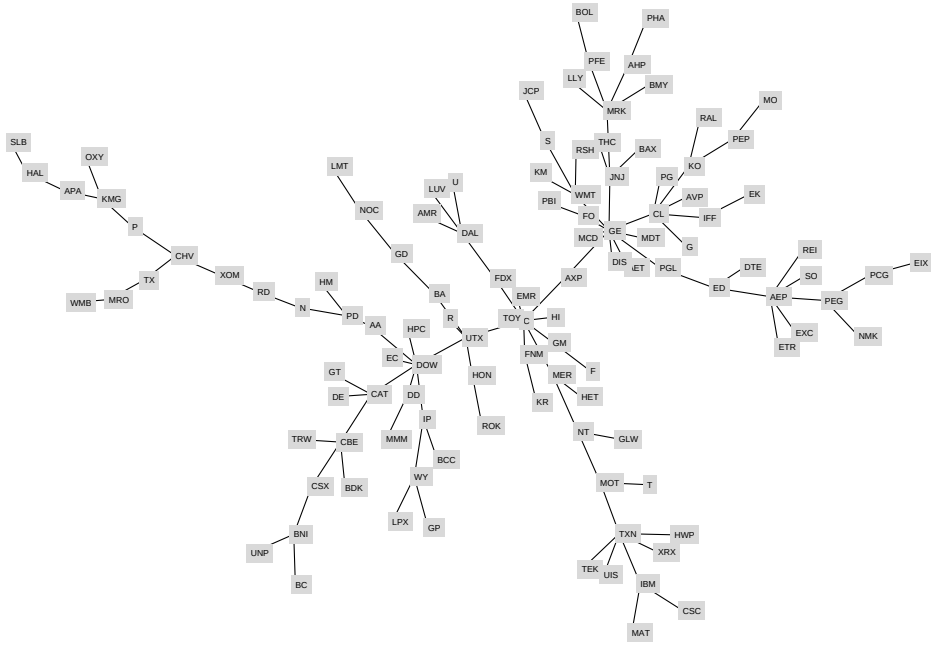

Figure 3.7: Maximal spanning tree of 116 NYSE traded stocks.

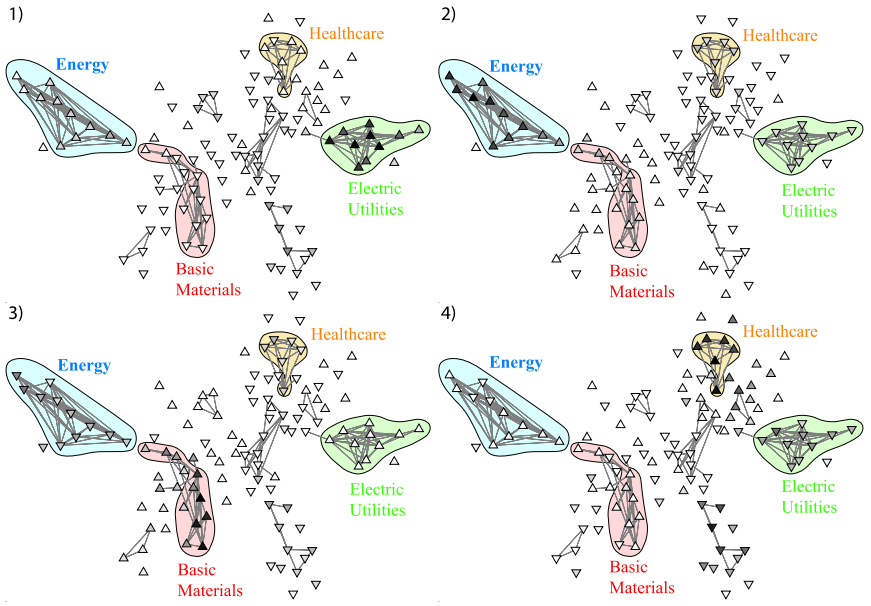

Figure 3.8: The asset graph for occupation $p = 0.025$ and the localisation of eigenvectors x_2, x_3, x_4 and x_5, where the indices denote ranks (panels 1-4, respectively). Modules corresponding to these eigenvectors, identified by the clique percolation method, are denoted by the shaded background. The orientation of the triangle at a node denotes the sign of the corresponding eigenvector component.The color of a node denotes the contribution of the corresponding component of x_1 to the length of the eigenvector. The largest component is colored black. For other nodes, linear scale is used such that white color indicates zero contribution.

measures to quantify the time stability of a network and showed that the basic structure of asset trees is very robust with respect to time. In Publication IV, this work is extended to cover asset graphs and results for non-denoised and denoised correlation matrices are compared. Interestingly, it turns out that the non-denoised matrices are more stable in time than the denoised ones. The modular structure can be fairly easily observed in both cases as already suggested in Publication II. However, it seems to be clearer in networks based on non-denoised matrices (see Fig. 3.9). In Publication II it is also shown that regardless of whether the correlation matrix is denoised[6] or not, there is no critical value of the thresholding parameter p_c that would reveal most of the modular structure. Therefore, from the point of view of module detection, noise reduction does not seem to be a key issue.

3.4.2 Full network approach

When analysing financial correlation matrices as full weighted networks, one usually first transforms them to weight matrices representing simple undirected weighted networks by defining the off-diagonal elements as $\boldsymbol{W}_{ij} = |\boldsymbol{C}_{ij}|$ and setting $\boldsymbol{W}_{ii} = 0$. The effect of this transformation on the modular structure of companies is shown to be small in (112) and the transformation is motivated by the fact that many network characteristics are defined for positive link weights only. From the point of view of network theory, it can be justified by interpreting the absolute values as measures of interaction strength without considering whether the interaction is positive or negative. Due to the market trend, usually only very few elements of $\boldsymbol{C}$ are negative and in most cases these exceptions have very small absolute values[7]. Therefore, the above transformation has usually no effect at all on the maximal spanning tree nor on asset graphs constructed with small or intermediate values of p. The spectrum of the matrix is shifted to the left by unity, but otherwise it can be expected that the numerical values of the spectral quantities do not significantly change, as stated in Publication II.

In Publication III, the modular structure of a full weighted network constructed in this way is analyzed with the RB-method and another very similar method by Arenas *et al.* (92). The modular structure produced by these methods corresponds very well with the Forbes classification of business sectors (113) as well as with the maximal spanning tree of the network (see Fig. 3.10). This is very encouraging, since the theoretical results derived in Publication III suggest that the RB-method may turn out to be a valuable tool in the analysis of dense weighted networks.

[6] In Publication II the denoising method suggested by Kim *et al.* (32) (see section the last paragraph in section 2.5) was used.

[7] When this is not the case, one should rather use the real values of the correlation coefficients instead, although some network characteristics cannot then be straightforwardly applied.

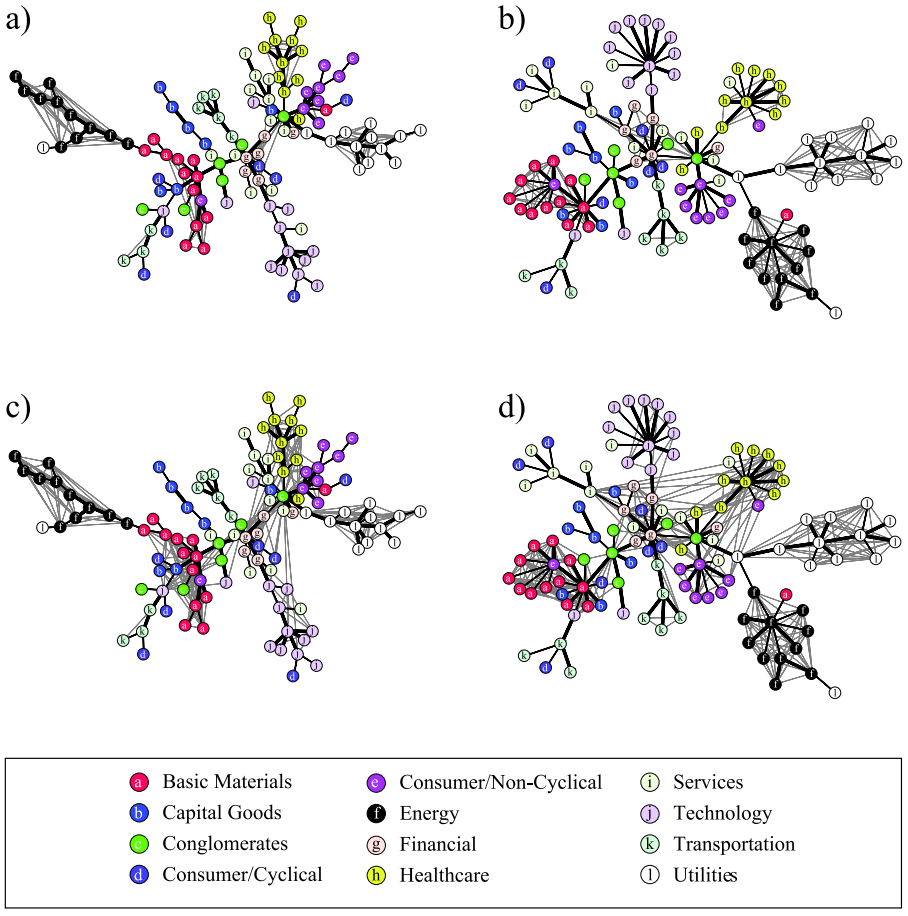

Figure 3.9: The MSTs based on a full weighted network of asset returns (left panels) and the corresponding denoised network (right panels) together with the asset graphs for $p = 0.025$ (upper panels) and $p = 0.05$ (lower panels), shown with the Forbes classification of stocks (113). The asset graphs are depicted with gray links and the MSTs with black. Links appearing in both MSTs are bolded.

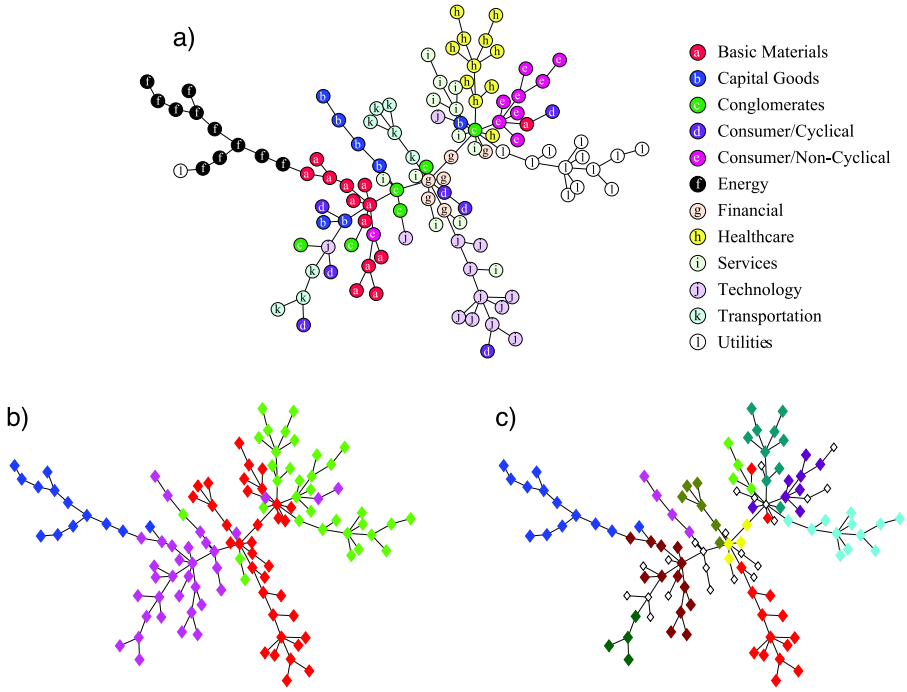

Figure 3.10: (a) The maximal spanning tree and business sectors according to Forbes (113). (b) The maximal spanning tree and the modular structure for $\gamma = 1$. Each color corresponds to a module. (c) The maximal spanning tree and the modular structure for $\gamma = 1.4$. Modules of size larger than two are depicted by different colors and the rest of the nodes by empty symbols.

Chapter 4

Summary of results and discussion

This chapter summarizes the publication part of this thesis and discusses future aspects of the field.

4.1 Summary of results

The MST, asset graph and spectral based approaches to the analysis of financial correlation matrices are compared for the first time in Publication I. It is shown that the eigenvector corresponding to the highest eigenvalue, *i.e.*, the market eigenvector is localised to the "back bone" of the MST, meaning that the most central nodes of the tree contribute most to the length of this eigenvector. The next highest ranking eigenvectors are localised to branches of the MST and correspond well to the modules observed with the thresholding analysis and the clique percolation method.

In Publication II, the study of the spectral properties of financial correlation matrices is continued. In this work, the network point of view is emphasized such that the correlation matrix is transformed to a weight matrix of a simple undirected weighted network before the analysis. It is shown both empirically and theoretically that the highest eigenvalue is practically proportional to the mean link weight and that the components of the corresponding eigenvector are well approximated by the node strengths. The properties of the next highest ranking eigenvectors are studied in detail with the conclusion that even though they are related to modules of strongly correlated companies, the extraction of the modular structure from these eigenvectors is a too formidable task. A very interesting observation is that the "borderline" between the random and non-random parts of the spectrum is very fuzzy. This means that there is no straightforward way to determine the information carrying eigenpairs, which is very worrisome for those

applying random matrix denoising. Very similar conclusions are made related to the diffusion based approach.

The study of asset graphs is also continued in Publication II. Asset graphs constructed using the denoising procedure suggested by Kim *et al.* are compared with the ordinary asset graphs and it is concluded that the denoising has very little effect on the modular structure of these graphs. This work is continued in Publication IV, but this time the standard random matrix denoising is applied. It is shown that the correspondence between business sectors and the modular structure of the networks based on the denoised correlation matrices is rather high, but not as high as without the denoising. Another very interesting observation made in Publication IV through a thorough network based analysis is that the non-denoised correlation matrices are more stable in time than the denoised ones.

In Publication III, the module detection method by Reichardt and Bornholdt is analysed in the context of dense weighted networks. A weighted random reference network is derived and the properties of the method are analysed with simple example networks. It is shown that for sparse weighted networks, the resolution of the weighted method behaves similarily to the resolution of the unweighted one. However, with large dense weighted networks the behaviour is very different; the resolution of the method is independent of the network size. In addition, the analysis of a full correlation-based financial network shows that the method is able to capture the modular structure of a dense real-world weighted network very well.

4.2 Discussion

The correlations between asset returns serve as main inputs in portfolio optimization, almost regardless of the method applied. Therefore, it is not surprising that financial institutions have shown a lot of interest in the ideas presented in this thesis, as indicated by various discussions between the author and financial practitioners. Currently, the industry practice is to classify stocks based on the business sectors or industries in which they operate. The investment rules or guidelines of *e.g.* mutual funds, meant to characterize the risk profile of the fund, are often built on these classification schemes. From the point of view of risk management, however, such classifications should naturally reflect the real modular structure of the markets and therefore they should be based on market data rather than third party observations. At least the traditional classifications should always be compared with the structure seen in the data. Related to this, another area in which the ideas presented in this thesis can be useful is the visualization of the market dependencies. The number of correlation coefficients scales as the square of the number of companies, which makes the visualization very challenging. The combination of MSTs and asset graphs, used throughout this thesis, has proven to be a very efficient and easy-to-understand way to do this.

Naturally, the methods and ideas presented in this work can also be applied in other fields of science. In the case of time series data, the methods can often be straightforwardly applied regardless of the exact nature of the time series. In other cases, one may first need to construct a network with some relevant case-dependent way. After this, however, the methods can easily be applied. The author himself has been involved in a project in which populations of the marine angiosperm *Posedonia oceanica* are studied with distance-based minimal spanning trees and thresholding analysis. From the point of view of this project, the financial data served as a great "test-bed" for the development and analysis of the methods used.

Discussion between the author and a quantitative trader from a hedge fund revealed that some financial practitioners might be interested in financial networks with a different approach to link weights. Indeed, there is no reason why we should restrict ourselves to equal-time correlations. For instance, a *cointegration*-based approach would probably capture information that is not present in the simple correlations.

Lastly, I wish to mention the ever-lasting fundamental debate about the existence of correlations between financial quantities. This is one of my favourite topics in quantitative finance, but since it is also a very difficult one, it is not discussed thoroughly in this work. However, I urge you all to think about the following quotation, before rereading this thesis. For those interested, I pretty much agree with it.

> ”I was speaking to an ex-academic recently about the subject of correlation between financial quantities. I said I didn’t believe in the existence of correlation in finance. The pompous old fool laughed to himself (you know the type) and said that, of course, it exists. Being uncharacteristically polite, I didn’t say what was on my mind, that you can measure correlation between being born in the year of rat and becoming a lawyer, but that may not mean much. Correlation doesn’t exist; deal with it.”
>
> Paul Wilmott (15)

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